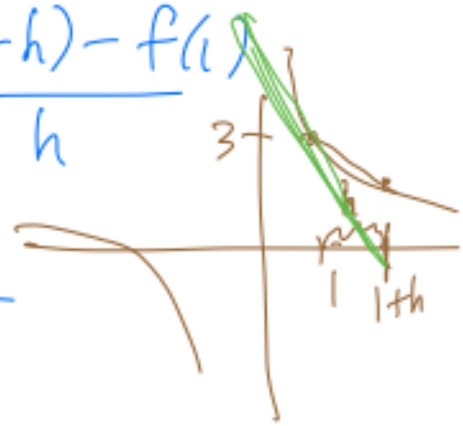
Example - find the equation of the tangent line to $y = \frac{x+2}{x}$ at $(1, 3)$.

$f(x)$ $y = 1 + \frac{2}{x}$

x y



$$\text{tangent slope} = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(1+h)+2}{1+h} - 3}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3+h}{1+h} - \frac{3(1+h)}{1+h}}{h}$$

$3+h-3-3h$

$$= \lim_{h\to 0} \frac{\frac{3+h-3(1+h)}{1+h}}{h/1}$$

$$= \lim_{h\to 0} \frac{\frac{-2h}{1+h}}{h/1} = \lim_{h\to 0} \frac{-2\cancel{h}}{(1+h)\cancel{h}}$$

$$= \boxed{-2}.$$

$$y - 3 = -2(x-1) \Rightarrow \boxed{y = -2x + 5}$$

Example. Using a limit of slopes, find the slope of the tangent line to $y = \frac{2^x}{4} - \sqrt{x}$ at $x = 4$. $(4,2)$

$f(4) = $ $y = \frac{2^4}{4} - \sqrt{4} = \frac{16}{4} - 2 = 4 - 2 = 2$

$$\text{tangent slope} = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

2

$$= \lim_{h \to 0} \frac{\left(\frac{2^{4+h}}{4} - \sqrt{4+h}\right) - 2}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\overset{16}{2^4} \cdot 2^h}{4} - \sqrt{4+h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{4 \cdot 2^h - \sqrt{4+h} - 2}{h}$$

$$\frac{AB}{A} = B \qquad \frac{A+B}{A} = \frac{A}{A} + \frac{B}{A}$$

$$= \lim_{h \to 0} \frac{4\left((2^h - 1) + 1\right) - \sqrt{4+h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{4(2^h - 1) + 4 - \sqrt{4+h} - 2}{h}$$

$$= \lim_{h\to 0} \frac{4(2^h - 1) + 2 - \sqrt{4+h}}{h}$$

$$= \lim_{h\to 0} \frac{4(2^h-1)}{h} + \lim_{h\to 0} \frac{2-\sqrt{4+h}}{h}$$

$$= \lim_{h\to 0} \frac{4\left(\left(e^{\ln 2}\right)^h - 1\right)}{h} + \lim_{h\to 0} \frac{(2-\sqrt{4+h})(2+\sqrt{4+h})}{h\,(2+\sqrt{4+h})}$$

$$= \lim_{h\to 0} 4\underbrace{\frac{\left(e^{(\ln 2)h} - 1\right)}{\underbrace{h(\ln 2)}_{\to 0}}}_{\to 1}(\ln 2) + \lim_{h\to 0} \frac{\overset{4-4+h}{4-(4+h)}}{h(2+\sqrt{4+h})}$$

$$= 4(\ln 2) + \lim_{h\to 0} \frac{-\cancel{h}^{\,1}}{\cancel{h}(2+\sqrt{4+h})}$$

$$= \boxed{4(\ln 2) - \frac{1}{4}} \text{ slope.}$$

tangent line

$$y - 2 = \left(4(\ln 2) - \tfrac{1}{4}\right)(x-4)$$

$y = \left(4(\ln 2) - \frac{1}{4}\right)x - 4\left(4(\ln 2) - \frac{1}{4}\right) + 2$

$y = \boxed{\left(4(\ln 2) - \frac{1}{4}\right)x - 16(\ln 2) + 3}$ $+1$

Application function $\approx$ tangent line near $x = 4$.

Check. function $\dfrac{2^{4.01}}{4} - \sqrt{4.01} =$ $\underline{2.0253}237607769$

(tangent line) $\left(4(\ln 2) - \frac{1}{4}\right)(4.01) - 16(\ln 2) + 3 =$

$\underline{2.025}225887$

pretty accurate.

This # for tangent slope is the instantaneous rate of change of the function

$\dfrac{f(4.01) - f(4)}{.01} = \dfrac{\text{change in } f}{\text{change in } x \quad .01}$

Definition. If $f : U \to \mathbb{R}$ is a function and $a \in U$, the derivative of f at a is denoted

interval

$$f'(a) \text{ or } \frac{df}{dx}(a) \text{ and is}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(it's possible it doesn't exist.)

Interpretation:

① $f'(a)$ = instantaneous rate of change of $f(x)$ as x increases from $x = a$.

② $f'(a)$ = slope of the tangent line to $y = f(x)$ at $(x, y) = (a, f(a))$.

The derivative function $f'(x)$ is the value of the derivative at every x in the domain.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Start with one function $f(x)$

$\rightarrow$ get new function $f'(x)$

that is the instantaneous rate of change of $f(x)$ at every point.

Example: Using the definition of derivative, find the derivative of $g(x) = x^3 - \frac{1}{x}$ at a general point $x = a$.

Solution:

$$g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \lim_{h\to 0} \frac{\left((a+h)^3 - \frac{1}{(a+h)}\right) - \left(a^3 - \frac{1}{a}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 - \frac{1}{a+h} - \cancel{a^3} + \frac{1}{a}}{h}$$

1

1 1

1 2 1

1 3 3 1

$$= \lim_{h \to 0} \frac{3a^2h + 3ah^2 + h^3 - \frac{1}{a+h} + \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} 3a^2 + \underbrace{3ah}_{\to 0} + \underbrace{h^2}_{\to 0} + \frac{-\frac{1}{a+h} + \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} 3a^2 + \frac{\frac{-a}{a(a+h)} + \frac{(a+h)}{a(a+h)}}{h}$$

$$= \lim_{h \to 0} 3a^2 + \frac{(\cancel{-a} + \cancel{a} + \cancel{h})}{a(a+h)} \cdot \frac{1}{\cancel{h}}$$

$$= \boxed{3a^2 + \frac{1}{a^2}}$$